

# The first encounter of two billiard particles of small radius

Dmitry Dolgopyat, Péter Nándori

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## Abstract

We prove that the time of the first collision between two particles in a Sinai billiard table converges weakly to an exponential distribution when time is rescaled by the inverse of the radius of the particles. This results provides a first step in studying the energy evolution of hard ball systems in the rare interaction limit.

## 1 Result

Understanding derivation of macroscopic laws from deterministic microscopic dynamics is an outstanding problem in mathematical physics. So far this has been achieved in a very limited number of cases. One prominent example, where this program has been implemented is the ideal gas of particles moving in dispersive domain (see [4, 15] and references therein). Unfortunately, the results for the ideal gas do not conform to the predictions of statistical mechanics. The reason is that in the ideal gas there is no mechanisms of coming to equilibrium, due to the lack of interaction between the particles. One way to rectify this situation is to study a rare interaction limit of a many particle system. One step in this direction is to understand how often the noninteracting particles are coming close to each other. This is the question studied in this paper.

Let  $\mathcal{D} = \mathbb{T}^2 \setminus \cup_{j=1}^J B_j$ , where  $B_j$  are disjoint strictly convex sets with  $\mathcal{C}^3$  smooth boundaries. The phase space of one billiard particle is  $\Omega = \mathcal{D} \times \mathcal{S}^1$ . The billiard flow consists of free flight among the scatterers and specular reflection off their boundaries and is denoted by  $\Phi^t : \Omega \rightarrow \Omega$ ,  $t \in \mathbb{R}$ . The phase space of the billiard ball map is  $\mathcal{M} = \{(q, v) \in \Omega : q \in \partial\mathcal{D}, \langle n, v \rangle \geq 0\}$ , where  $v$  is the normal vector of  $\partial\mathcal{D}$  at  $q$  pointing into  $\mathcal{D}$ . The billiard ball map  $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$  takes the particle from one collision to the next one. The flow time between two collisions is bounded from below by  $\tau_{\min} > 0$  and we also assume that it is

bounded from above by  $\tau_{\max} < \infty$  (this is called the finite horizon condition). The invariant measure of the billiard flow is  $\mu = c_\mu dq dv$  and that of the billiard map is  $\nu = c_\nu \cos(\varphi, n) d\varphi dr$ , where  $c_\mu = \frac{1}{2\pi|\mathcal{D}|}$  and  $c_\nu = \frac{1}{2|\partial\mathcal{D}|}$ . We will denote by  $\Pi_q$  the projection from  $\Omega$  to  $\mathcal{D}$  and similarly by  $\Pi_\varphi$  the projection from  $\Omega$  to  $\mathcal{S}^1$ .

Let us consider two billiard particles on the same domain, i.e.  $(q_i, \varphi_i) \in \Omega$  for  $i = 1, 2$ . Without loss of generality, we assume that the first particle travels with speed one and the second one travels with speed  $\lambda \in [0, 1]$  (note that by rescaling time we can fix the speed of the faster one). Our main objective is to estimate the time we need to wait until the two particles get  $\varepsilon$  close. Thus we introduce the notation

$$\mathcal{A}_{\lambda, t, \varepsilon} = \{(q_1, \varphi_1), (q_2, \varphi_2) : \exists s \in [0, t] : \|\Pi_q \Phi^s(q_1, \varphi_1) - \Pi_q \Phi^{\lambda s}(q_2, \varphi_2)\| \leq \varepsilon\}.$$

Next, we define the rate function  $\rho$  by

$$\rho(\lambda) = \frac{1}{2\pi|\mathcal{D}|} \int_0^{2\pi} \sqrt{1 - 2\lambda \cos \varphi + \lambda^2} d\varphi. \quad (1)$$

Note that  $\rho$  is bounded away from zero. Our main theorem is the following

**Theorem 1.** *For Lebesgue-a.e.  $\lambda \in [0, 1]$  and for every fixed  $T$ ,*

$$\lim_{\varepsilon \rightarrow 0} (\mu \times \mu)(\mathcal{A}_{\lambda, T/\varepsilon, \varepsilon}) = 1 - e^{-\rho(\lambda)T}.$$

**Remark 2.** *Theorem 1 was conjectured in an unpublished paper by Thomas Gilbert [12]. In particular, he computed the function  $\rho$  based on the ergodicity and explicit formulas for the mean free path (a computation along the lines of [2]).*

## 2 Preliminaries

We introduce the basic definitions and lemmas that we will need to prove Theorem 1. Fix some constant  $\mathfrak{s} < \tau_{\min}$  (e.g.  $\mathfrak{s} = \tau_{\min}/2$ ). In this section, we study  $\Phi^{\mathfrak{s}}$ , the time  $\mathfrak{s}$ -map of the billiard flow (that is, we only study one particle). The forthcoming definitions and statements concerning the time  $\mathfrak{s}$ -map are very similar to the corresponding definitions and statements for the billiard map  $\mathcal{F}$ . As the proofs are also very similar, we do not give detailed proofs here, instead we highlight the differences and strongly encourage the reader to consult the detailed exposition of [9].

The phase space of  $\Phi^{\mathfrak{s}}$  is  $\Omega$ . Let us consider the Jacobi coordinates in the tangent space  $\mathcal{T}_X\Omega$  at the point  $X = (x_1, x_2, \varphi)$ :

$$d\eta = \cos \varphi dx_1 + \sin \varphi dx_2, \quad d\xi = -\sin \varphi dx_1 + \cos \varphi dx_2, \quad d\omega = d\varphi.$$

For  $X = (q, \varphi) \in \Omega$  denote by  $\mathcal{T}_X^\perp$  the subspace spanned by  $(d\xi, d\omega)$ . Then for a.e.  $X$  there exists a 1 dimensional unstable and a 1 dimensional stable manifold  $W^u(X), W^s(X)$ , both of which satisfy  $\mathcal{T}_X W^{u/s} \in \mathcal{T}_X^\perp \Omega$ . These coincide with the stable and unstable manifolds of the flow, see Section 6.8 in [9].

Next, we want to introduce an extension of the class of the unstable manifolds, namely the unstable curves. First we need to recall some notations from [9]. The first collision in negative time is  $t^-(X)$ , that is

$$t^-(X) = \max\{t < 0 : \Phi^t(X) \in \mathcal{M}\}.$$

Next we define the projection  $\mathbf{P}^-(X) = \Phi^{t^-(X)}(X)$  from  $\Omega$  to  $\mathcal{M}$ . Then the linear map

$$D\mathbf{P}^-(X) : \mathcal{T}_X^\perp \Omega \rightarrow \mathcal{T}_{t^-(X)} \mathcal{M}$$

is a bijection. The unstable cone field of the map  $\mathcal{F}$  is constructed in Section 4.4 of [9]: for any  $x \in \mathcal{M}$ ,

$$\hat{\mathcal{C}}_x = \{(dr, d\varphi) \in \mathcal{T}_x \mathcal{M} : \mathcal{K} \leq d\varphi/dr \leq \mathcal{K} + \cos \varphi / t^-(x)\},$$

where  $\mathcal{K}$  is the curvature of  $\partial\mathcal{D}$  at the configurational component of  $x$ . Now we extend this cone field to  $\mathcal{T}\Omega$  by

$$\hat{\mathcal{C}}_X = (D\mathbf{P}^-(X))^{-1} \hat{\mathcal{C}}_{\mathbf{P}^-(X)} \subset \mathcal{T}_X^\perp \Omega.$$

Clearly,  $\hat{\mathcal{C}}_X$  is invariant under the flow in the usual sense. First, we call a curve  $W \subset \Omega$  *contact unstable curve* if at every point  $X \in W$ , the tangent line  $\mathcal{T}_X W$  belongs to the cone  $\hat{\mathcal{C}}_X$ . Although the contact unstable curves would suffice for the sake of the present work, we prefer to slightly generalize the concept. Thus we introduce

$$\mathcal{C}_X = \{(d\eta, d\xi, d\omega) \in \mathcal{T}_X \Omega : (d\xi, d\omega) \in \hat{\mathcal{C}}_X, d\eta/d\xi < C_f\},$$

where  $C_f$  is a fixed universal constant ( $f$  stands for flow). Also note that for tangent vectors in  $\mathcal{C}_X$ ,  $d\eta/d\omega < C'_f$  holds with some universal constant  $C'_f$ . We say that  $W \subset \Omega$  is an *unstable curve* if at every point  $X \in W$ ,  $\mathcal{T}_X W \in \mathcal{C}_X$ . The image of an unstable curve is unstable. Furthermore, unstable curves are stretched by the map  $\Phi^s$  in the sense that

$$\mathcal{J}_W(\Phi^s)^n(X) := \frac{\|D_X(\Phi^s)^n(dX)\|}{\|dX\|} \geq C\Lambda^n \quad (2)$$

with some  $\Lambda = \Lambda(C_f) > 1$  uniformly in  $X$  and  $dX \in \mathcal{C}_X$ . Here,  $\mathcal{J}$  stands for Jacobian and  $\Lambda$  is a constant that only depends on  $\mathcal{D}$  and  $C_f$  (it also depends on  $\mathfrak{s}$  but we ignore this dependence here since we have chosen  $\mathfrak{s} = \tau_{min}/2$ ).

The expansion is bounded from below but it is unbounded from above near grazing collisions. So as to recover distortion bounds and following the

common approach introduced by Bunimovich, Chernov and Sinai ([5]), we decompose the phase space  $\Omega$  into homogeneity domains  $\mathbf{G}, \mathbf{H}_k$ , with  $k = 0$  or  $|k| \geq k_0$ . Namely, for such  $k$ 's, we define

$$\mathbf{H}_k = \{X \in \Omega : t^-(X) < \mathfrak{s} \text{ and } \mathbf{P}^-(X) \in \mathbb{H}_k\}$$

where

$$\mathbb{H}_k = \begin{cases} \{(r, \varphi) : -\pi/2 + k_0^2 < \varphi < +\pi/2 - k_0^2\} & \text{for } k = 0 \\ \{(r, \varphi) : \pi/2 - k^2 < \varphi < +\pi/2 - (k+1)^2\} & \text{for } k \geq k_0 \\ \{(r, \varphi) : -\pi/2 + (k+1)^2 < \varphi < -\pi/2 + k^2\} & \text{for } k \leq -k_0 \end{cases}$$

are the usual homogeneity strips of the map  $\mathcal{F}$ . Finally, we define  $\mathbf{G} = \{X \in \Omega : t^-(X) > \mathfrak{s}\}$ . We say that an unstable curve  $W$  is *weakly homogeneous*, if it belongs to one homogeneity domain. Then we have the following distortion bound (cf. Lemma 5.27 in [9]): if  $(\Phi^s)^{-n}W$  is weakly homogeneous for every  $0 \leq n \leq N-1$ , then for all  $1 \leq n \leq N$  and for all  $Y, Z \in W$ ,

$$e^{-C_d \frac{|W(Y,Z)|}{|W|^{2/3}}} \leq \frac{\mathcal{J}_W(\Phi^s)^{-n}(Y)}{\mathcal{J}_W(\Phi^s)^{-n}(Z)} \leq e^{C_d \frac{|W(Y,Z)|}{|W|^{2/3}}} \quad (3)$$

with some constant  $C_d$  depending on  $\mathcal{D}$  and  $C_f$ . Here,  $|W(Y, Z)|$  is the length of the segment of  $W$  lying between  $Y$  and  $Z$ . The proof of (3) is analogous to that of Lemma 5.27 in [9] (observe that  $\frac{d}{dX} \ln \mathcal{J}_{\tilde{W}}(\Phi^s)^{-1}(\tilde{X})$  is bounded if  $\tilde{W} \subset \mathbf{G}$ , other cases are analogous to (5.8) in [9]).

Next, we define *standard pairs* for the map  $\Phi^s$ . A standard pair  $\ell = (W, \rho)$  consist of a weakly homogeneous unstable curve  $W$  and a probability density  $\rho$  supported on  $W$  which satisfies

$$\left| \ln \frac{d\rho}{dLeb}(X) - \ln \frac{d\rho}{dLeb}(Y) \right| \leq C_r \frac{|W(X, Y)|}{|W|^{2/3}} \quad (4)$$

where  $C_r$  is some fixed big constant.

Now we are ready to state a key Lemma, which is often called the growth lemma.

**Lemma 1.** *Let  $\ell = (W, \rho)$  be a standard pair and  $A$  a measurable set. Then*

$$\mathbb{E}_\ell(A \circ (\Phi^s)^n) = \sum_a c_{a,n} \mathbb{E}_{\ell_{an}}(A), \quad (5)$$

where  $c_{a,n} > 0$ ,  $\sum_a c_{a,n} = 1$ ;  $\ell_{an} = (W_{an}, \rho_{an})$  are standard pairs such that  $\cup_a W_{an} = (\Phi^s)^n W$  and  $\rho_{an}$  is the push-forward of  $\rho$  by  $(\Phi^s)^n$  up to a multiplicative constant. Finally, there are constants  $\varkappa, C_1$  (depending on  $\mathcal{D}$  and  $C_f$ ), such that if  $n > \varkappa |\log \text{length}(W)|$ , then

$$\sum_{\text{length}(\ell_{an}) < \varepsilon} c_{a,n} < C_1 \varepsilon. \quad (6)$$

To fix terminology, we will call (5) a Markov decomposition.

The difference between the proof of Lemma 1 and analogous lemmas for  $\mathcal{F}$  (Section 5.10 and 7.4 in [9]) is slightly more substantial than in case of the previous statements. Namely, the proof in [9] for the case of  $\mathcal{F}$  is based on the one-step expansion estimate

$$\liminf_{\delta \rightarrow 0} \sup_{\mathcal{W}: |\mathcal{W}| < \delta} \sum_i \lambda(\mathcal{W}_i) < 1, \quad (7)$$

where  $\mathcal{W}$  is an unstable curve for the map  $\mathcal{F}$ ,  $\mathcal{W}_i$  are the H-components of  $\mathcal{F}(\mathcal{W})$  and  $\lambda(\mathcal{W}_i)$  is the maximal contraction factor of  $\mathcal{F}^{-1}$  on  $\mathcal{W}_i$ . (These have similar definitions to ours, see [9]). We note that (7) cannot hold for our case. Indeed, if  $W$  is such an unstable curve that some of its points experience a near perpendicular collision within time  $\mathfrak{s}$  with a scatterer of small curvature and some other points do not collide within time  $\mathfrak{s}$ , then the one-step expansion is violated (even with an adapted norm). That is why we prove the  $N$ -step expansion instead (with some suitable  $N$ ):

$$\liminf_{\delta \rightarrow 0} \sup_{W: |W| < \delta} \sum_i \lambda(W_{i,N}) < 1, \quad (8)$$

where  $W_{i,N}$  are the H-components of  $(\Phi^{\mathfrak{s}})^N W$  and  $\lambda(W_{i,N})$  is the maximal contraction factor of  $(\Phi^{\mathfrak{s}})^{-N}$  on  $W_{i,N}$ . We note that similar  $N$ -step expansions have been used several times, e.g. in case of billiards with corner points [7, 10].

The proof of (8) relies on a complexity estimate which we derive next. First, we write  $\Omega = \mathbf{H} \cup \mathbf{G}$  (modulo a set of zero  $\mu$  measure), where  $\mathbf{H} = \{X \in \Omega : t^-(X) < \mathfrak{s}\}$  (and consequently  $\overline{\mathbf{H}} = \cup \overline{\mathbf{H}_k}$ ). Note that  $\Phi^{\mathfrak{s}}$  is continuous on the domains  $\mathbf{H}$  and  $\mathbf{G}$ . For an unstable curve  $W$  we say that a sequence  $(A_1, \dots, A_n) \in \{\mathbf{H}, \mathbf{G}\}^n$  is admissible if there is some point  $X \in W$  such that  $(\Phi^{\mathfrak{s}})^i \in A_i$  for all  $i = 1, \dots, n$ . Next, we define

$$K_n(\delta) = \max_{W: |W| < \delta} \#\{\text{admissible sequences } (A_1, \dots, A_n)\}.$$

Now our complexity bound is the following. There exists some  $L < \infty$  such that for all  $n > 0$ ,

$$\lim_{\delta \rightarrow \infty} K_n(\delta) < Ln^2. \quad (9)$$

In order to derive (9), we first recall the complexity bound of the map  $\mathcal{F}$  from [5]. The complexity  $\mathcal{K}_n(\delta)$  of the map  $\mathcal{F}$  (that is the maximal number the singularity set  $\{\varphi = \pm\pi/2\}$  can cut an unstable curve of length  $\leq \delta$  during  $n$  iteration) satisfies

$$\lim_{\delta \rightarrow \infty} \mathcal{K}_n(\delta) < An. \quad (10)$$

There is two reasons why  $(\Phi^{\mathfrak{s}})^n$  can cut an unstable curve: grazing collisions and collisions at times which are integer multiples of  $\mathfrak{s}$ . The grazing collisions

are liable for the fragmentation of unstable curves of the map  $\mathcal{F}$ , hence can be bounded by (10). Namely, for fixed  $n$  and for small enough  $\delta$ ,  $W$  can be cut into pieces  $W_\alpha$ ,  $\alpha \leq Bn$  such that all  $X, Y$  belonging to the same piece  $W_\alpha$  collide on the same sequence of scatterers during flow time  $n\mathfrak{s}$ . Indeed, as the number of collisions during flow time  $n\mathfrak{s}$  is bounded by  $n\mathfrak{s}/\tau_{\min}$ , we can choose  $B = A\mathfrak{s}/\tau_{\min}$ . Now pick some  $W_\alpha$ . Observe that by definition,  $\Pi_q(\Phi^{\mathfrak{s}})^i W_\alpha$ , the projection of the unstable curve  $(\Phi^{\mathfrak{s}})^i W_\alpha$  to the configuration space, is convex. Consequently there can be at most 2 points on  $W_\alpha$  which collide exactly at time  $(i+1)\mathfrak{s}$ . We conclude that each  $W_\alpha$  is cut into at most  $2n+1$  pieces which proves (9) (with say  $L = 3A$ ).

The derivation of (8) from (9) goes along the lines of the proof of Lemma 5.56 in [9]. (We need to choose  $N$  such that  $LN^3 < \Lambda^N$  so as to bound  $\sum \lambda(W_{i,N})$  for  $i$ 's which never visited the nearly grazing domains  $\mathbf{H}_k$  with  $|k| \geq k_0$ , and choose  $k_0$  big to bound the remaining terms.) Finally, the proof of Lemma 1 based on (8) is again similar to the usual argument (see also [7, 10]).

For standard pairs  $\ell = (W, \rho)$ , we will write  $\mathbb{E}_\ell$  for the integral with respect to  $\rho$  and  $\mathbb{P}_\ell(A) = \mathbb{E}_\ell(1_A)$ . A standard family is a weighted average of standard pairs:  $\mathcal{G} = (W_\alpha, \rho_\alpha)$ ,  $\alpha \in \mathfrak{A}$  and a measure  $\lambda_\mathcal{G}$  on the (possibly infinite) index set  $\mathfrak{A}$ . The  $\mathcal{Z}$ -function of  $\mathcal{G}$  is defined by

$$\mathcal{Z}_\mathcal{G} = \sup_{\varepsilon > 0} \frac{\int_{\mathfrak{A}} \mathbb{P}_{\ell_\alpha}(r_\mathcal{G} < \varepsilon) d\lambda_\mathcal{G}}{\varepsilon}.$$

Here  $r_\mathcal{G}(X)$  is the distance of  $X$  and the closest endpoint of  $W_\alpha$ , where  $W_\alpha \ni X$ . With these notations, Lemma 1 can be shortly reformulated by saying that  $\mathcal{G}_n$ , the image of a standard family  $\mathcal{G}$  under  $(\Phi^{\mathfrak{s}})^n$  is a standard family and there are constants  $\vartheta < 1$ ,  $\beta_1$  and  $\beta_2$  (depending on  $\mathcal{D}$  and  $C_f$ ) such that  $\mathcal{Z}_{\mathcal{G}_n} \leq \beta_1 \vartheta^n \mathcal{Z}_\mathcal{G} + \beta_2$ .

One very important example for a standard family is the decomposition of the SRB measure  $\mu$  to local unstable manifolds. The fact that the  $\mathcal{Z}$  function is finite is far from being obvious, but it follows from early works of Sinai. (See Theorem 5.17 in [9] for the case of  $\mathcal{F}$ ; our case is analogous).

We conclude this section with a stretched exponential bound on the decay of correlations with respect to a standard pair. First, we recall a few definitions from [8]. For some function  $F : \Omega \rightarrow \mathbb{R}$ ,  $x \in \Omega$  and  $r > 0$ , we write  $\text{osc}_r(F, x) = \sup_B F - \inf_B F$ , where  $B$  is the ball of radius  $r$  centered at  $x$ . We say that  $F$  is generalized Hölder continuous with exponent  $\alpha \in (0, 1]$  if

$$\|F\|_\alpha = \sup_r r^{-\alpha} \int_\Omega \text{osc}_r(F, x) d\mu(x) < \infty,$$

and write  $\text{var}_\alpha(F) = \|F\|_\alpha + \sup_\Omega F - \inf_\Omega F$ .

**Theorem 3.** *Let  $\ell = (W, \rho)$  be a standard pair and let  $F : \Omega \rightarrow \mathbb{R}$  be generalized Hölder continuous with parameter  $\alpha$  and  $\int_{\Omega} F d\mu = 0$ . Then*

$$|\mathbb{E}_{\ell}(F \circ \Phi^t)| \leq |W|^{-1} C \text{var}_{\alpha}(F) e^{-a\sqrt{t}},$$

*with constants  $C, \alpha$  depending only on  $\mathcal{D}$ ,  $C_f$  and  $\alpha$ .*

A sketch of proof of Theorem 3 was given by Chernov in [8] for contact unstable curves (a more detailed proof will be provided in [3]). For general unstable curves, one can apply the same smoothening as in Corollary 1.2 of [8] after chopping  $W$  to pieces of length  $\leq \varepsilon/C_f$ . We note that a recent paper [1] obtains exponential mixing for smooth observable. Using this results it seems likely that the bound of Theorem 3 can be improved to exponential. However, we do not pursue this question here since the bounds of [8] and [3] are sufficient for our purposes.

## 3 Proof of Theorem 1

### 3.1 Idea of the proof

We will say that a time  $t$  is microscopic if  $t \leq 1$ , mesoscopic if  $1 < t \leq \delta'/\varepsilon$  and macroscopic if  $\delta'/\varepsilon < t$ .

For most of the proof, we fix  $(q_1, \varphi_1)$  and use the results of Sinai billiards (cf. Section 2) for the second particle.

Recall that a random variable  $\mathbf{T}$  has exponential distribution with parameter  $\rho$  if

$$\mathbb{P}(\mathbf{T} \in [t, t + \delta'] | \mathbf{T} > t) = \rho \delta' (1 + o(1)). \quad (11)$$

Therefore in our setting we need to show that probability that the first close encounter happens during the time  $[\frac{t}{\varepsilon}, \frac{t+\delta'}{\varepsilon}]$  given that there was no collision in the past equals to  $\rho(\lambda) \delta' (1 + o(1))$ . More precisely, in order to ensure the near independence of consecutive intervals it is convenient to introduce short buffer zone between them. The fact that the first collision is unlikely to fall to a buffer zone then follows by Markov inequality. To estimate the collision probability during an interval of size  $\delta'/\varepsilon$  we divide it into intervals of length  $\delta \ll 1$ . Using elementary geometry we show that the measure of the trajectories having a close encounter during such an interval equals to  $\rho(\lambda)(1 + o(1))$ . Summing the probabilities along all short intervals in a given interval of size  $\delta'/\varepsilon$  we get  $\rho(\lambda) \delta'$ . Thus to obtain (11) for our system we need to show that recollisions have smaller order if  $\delta'$  is small. To prove this we first estimate a measure of trajectories having more than one small encounter and then use mixing to accommodate conditioning on the past. The recollisions happening during relatively separated times (that is if the collision times are at least  $\ln^{100} \varepsilon$  apart)

are ruled out by mixing, while fast recollisions are handled by a geometric argument. Namely we show that such recollisions are easily destroyed if we tune the speed of the second particle. This is the only part of the proof which does not work for **all** values of  $\lambda$ . At this step we also need to exclude the encounters where the particles either have almost parallel velocities or are close to the scatterers.

Specifically, we introduce

$$\begin{aligned}\mathcal{A}_{\lambda,t,\varepsilon}^\xi &= \{(q_1, \varphi_1), (q_2, \varphi_2) : \exists s \in [0, t] : \\ &\quad \|\Pi_q \Phi^s(q_1, \varphi_1) - \Pi_q \Phi^{s\lambda}(q_2, \varphi_2)\| \leq \varepsilon, \\ &\quad \text{dist}(\Pi_q \Phi^s(q_1, \varphi_1), \partial\mathcal{D}) > \xi, \\ &\quad |\Pi_\varphi \Phi^s(q_1, \varphi_1) - \Pi_\varphi \Phi^{s\lambda}(q_2, \varphi_2) \pmod{\pi}| > \xi\}.\end{aligned}$$

and

$$\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi(q_1, \varphi_1) = \{(q_2, \varphi_2) : (q_1, \varphi_1, q_2, \varphi_2) \in \mathcal{A}_{\lambda,\delta,\varepsilon}^\xi\}. \quad (12)$$

The order of choice of the parameters can be summarized as

$$\varepsilon \ll \delta \ll \delta' \ll \xi \ll 1$$

(for each inequality  $\ll$  we impose finitely many upper bounds along the way, it is possible to take the smallest one).

To simplify notation, we will say that the two particles have a "good collision" at time  $s$  if the last three lines of the definition of  $\mathcal{A}_{\lambda,t,\varepsilon}^\xi$  are true (note that this is not a real collision, and the notion is dependent on  $\lambda$ ,  $\varepsilon$  and  $\xi$ ).

**Remark 4.** *We note that the strategy of using (11) for proving exponential distribution for hitting times to small sets with relying on mixing to handle the mesoscopic return times and on geometry to handle short return times is by now pretty standard in the dynamical systems literature (see e.g. [6, 11, 13, 14] and reference therein), however the implementation depends very much on the system at hand.*

## 3.2 Microscopic and mesoscopic time

Our first lemma concerns microscopic timescales:

**Lemma 2.** *For all  $\lambda \in (0, 1]$ ,*

$$\lim_{\xi \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{1}{\delta} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mu \times \mu)(\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi) = \rho(\lambda)$$

*Proof.* Clearly, we can assume that  $\text{dist}(q_1, \partial\mathcal{D}) > 2\xi$  (which has  $\mu$ -measure  $1 - O(\xi)$ ). Then, if the two point particles collide within time  $\delta$ , then necessarily  $\text{dist}(q_2, \partial\mathcal{D}) > \xi$  which means that none of the particles collide with



the boundary of the billiard table within time  $\delta$  and in particular there can be no more than one binary collision. Now we fix  $q_1$  as above and w.l.o.g. write  $\varphi_1 = 0$ . Then we compute the two dimensional measure of the surface

$$\mathcal{A}_{\lambda,\delta,0}^\xi(q_1, 0) = \{(q_2, \varphi_2) : (q_1, 0, q_2, \varphi_2) \in \mathcal{A}_{\lambda,\delta,0}^\xi\} \subset \mathbb{R}^3. \quad (13)$$

The collision takes place at time  $t \in [0, \delta]$ . Now we have the following parametrization of this surface:

$$\begin{aligned} u : [0, \delta] \times I_\xi &\rightarrow \mathbb{R}^3 \text{ with } I_\xi = ([\xi, \pi - \xi] \cup [\pi + \xi, 2\pi - \xi]), \\ u(t, \varphi_2) &= (t - \lambda t \cos \varphi_2, \lambda t \sin \varphi_2, \varphi_2) + (q_1, 0) \end{aligned}$$

Then

$$\begin{aligned} &Leb_2(\mathcal{A}_{\lambda,\delta,0}^\xi(q_1, 0)) \\ &= \int_{t=0}^{\delta} \int_{\varphi_2 \in I_\xi} \sqrt{1 - 2\lambda \cos \varphi_2 + \lambda^2 + \lambda^2 t^2 \cos^2 \varphi_2 - 2\lambda^3 t^2 \cos \varphi_2 + \lambda^4 t^4} d\varphi_2 dt \\ &\sim \delta \int_{\varphi_2 \in I_\xi} \sqrt{1 - 2\lambda \cos \varphi_2 + \lambda^2} d\varphi_2 \end{aligned}$$

as  $\delta \rightarrow 0$ . Now the asymptotics for  $\varepsilon \rightarrow 0$

$$\begin{aligned} &\mu(q_2, \varphi_2 : (q_1, 0, q_2, \varphi_2) \in \mathcal{A}_{\lambda,\delta,\varepsilon}^\xi) \\ &= \frac{1}{2\pi|\mathcal{D}|} Leb_3(q_2, \varphi_2 : (q_1, 0, q_2, \varphi_2) \in \mathcal{A}_{\lambda,\delta,\varepsilon}^\xi) \\ &\sim \frac{1}{2\pi|\mathcal{D}|} \varepsilon Leb_2(\mathcal{A}_{\lambda,\delta,0}^\xi(q_1, 0)) \end{aligned}$$

completes the proof.  $\square$

The following more technical version of Lemma 2 also follows from the above proof.

**Lemma 3.** *For any  $\eta > 0$  there is some  $\xi_0$  such that for all  $\xi < \xi_0$  there is some  $\delta_0 = \delta_0(\eta, \xi)$  such that for all  $\delta < \delta_0$  there is some  $\varepsilon_0 = \varepsilon_0(\eta, \xi, \delta)$  such that for all  $\varepsilon < \varepsilon_0$*

$$(a) \text{ for all } q_1, \varphi_1, \mu\left(\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi(q_1, \varphi_1)\right) < (\rho(\lambda) + \eta)\delta\varepsilon.$$

$$(b) \text{ for all } q_1 \text{ with } \text{dist}(q_1, \partial\mathcal{D}) > 2\xi \text{ and for all } \varphi_1, \mu\left(\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi(q_1, \varphi_1)\right) > (\rho(\lambda) - \eta)\delta\varepsilon.$$

The next lemma bounds the probability of short return and is of crucial importance.

**Lemma 4.** For  $Leb_1$ -a.e.  $\lambda \in [0, 1]$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{(\mu \times \mu)(A_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi)}{\varepsilon^{1.99}} = 0$$

*Proof.* First observe that

$$A_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi \subset \cup_{k=0}^{\delta/\varepsilon} (\Phi^{-k\varepsilon} \times \Phi^{-k\lambda\varepsilon})(A_{\lambda, \varepsilon, \varepsilon}^\xi \cap (\Phi^{-\delta+k\varepsilon} \times \Phi^{\lambda(-\delta+k\varepsilon)})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi)$$

Using this, the invariance of  $\mu$  and the fact that two "good collisions" (in the sense defined after  $\mathcal{A}_{\lambda, t, \varepsilon}^\xi$ ) are necessarily separated by  $\xi$  we conclude

$$\begin{aligned} & (\mu \times \mu)(A_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) \\ & \leq \frac{\delta}{\varepsilon} (\mu \times \mu)(A_{\lambda, \varepsilon, \varepsilon}^\xi \cap (\Phi^{-\xi} \times \Phi^{-\lambda\xi})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) \\ & \leq \frac{\delta}{\varepsilon} (\mu \times \mu)(\{\|q_1 - q_2\| < 3\varepsilon\} \cap (\Phi^{-\xi} \times \Phi^{-\lambda\xi})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi). \end{aligned} \quad (14)$$

Next we prove that for arbitrary  $q_1, \varphi_1, q_2, \varphi_2$  fixed,

$$Leb_1(\lambda : (q_1, \varphi_1, q_2, \varphi_2) \in (\Phi^{-\xi} \times \Phi^{-\lambda\xi})A_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) < C_\xi \varepsilon \log^{200} \varepsilon. \quad (15)$$

Since, by the definition of a good collision, the particles come to a close encounter with transversal velocities, their paths should intersect near the time of a close encounter. We note that as the free flight is bounded from below, the trajectories in the configuration space  $\Pi_q\{\Phi^t(q_i, \varphi_i)\}_{t \in [\xi, \log^{100} \varepsilon]}$  can have at most  $\log^{200} \varepsilon$  "good" intersections (where good means that their angle is at least  $\xi$  and their distance from the boundary is at least  $\xi/2$ ). Let us denote the time instants when the first particle arrives at these intersections by  $t_i, i < \log^{200} \varepsilon$ . Now a simple geometry shows that  $(q_1, \varphi_1, q_2, \varphi_2) \in (\Phi^{-\xi} \times \Phi^{-\lambda\xi})\mathcal{A}_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi$  implies

$$\|\Pi_q \Phi^{t_k}(q_1, \varphi_1) - \Pi_q \Phi^{t_k}(q_2, \varphi_2)\| < \frac{2\varepsilon}{\sin \xi} \quad \text{for some } k < \log^{200} \varepsilon. \quad (16)$$

Now the set of  $\lambda$ 's satisfying (16) for a fixed  $k$  is an interval whose length is bounded by  $\frac{2\varepsilon}{t_k \sin \xi} < C_\xi \varepsilon$ . (15) follows.

Combining (14) and (15) we obtain

$$\begin{aligned} & \int_\lambda (\mu \times \mu)(\mathcal{A}_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})\mathcal{A}_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) dLeb_1(\lambda) \\ & \leq \frac{\delta}{\varepsilon} \int_\lambda \int_{\{\|q_1 - q_2\| < 3\varepsilon\}, \varphi_1, \varphi_2} 1_{(\Phi^{-\xi} \times \Phi^{-\lambda\xi})\mathcal{A}_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi} d(\mu \times \mu) dLeb_1(\lambda) \\ & \leq C_{\xi, \delta} \varepsilon^2 \log^{200} \varepsilon. \end{aligned}$$

The Markov inequality gives

$$Leb_1\{\lambda : (\mu \times \mu)(\mathcal{A}_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})\mathcal{A}_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) > \varepsilon^{1.995}\} < C_{\xi, \delta} \varepsilon^{0.005} \log^{200} \varepsilon \quad (17)$$

Clearly, (17) holds with  $\log^{100} \varepsilon$  replaced by  $\log^{100}(2\varepsilon)$  (possibly with some new constant  $C_{\xi, \delta}$ ). Thus we have for all positive integer  $l$ ,

$$Leb_1\{\lambda : \exists \varepsilon \in [2^{-l}, 2^{-l-1}]; (\mu \times \mu)(\mathcal{A}_{\lambda, \delta, \varepsilon}^\xi \cap (\Phi^{-\delta} \times \Phi^{-\lambda\delta})\mathcal{A}_{\lambda, \log^{100} \varepsilon, \varepsilon}^\xi) > \varepsilon^{1.995}\} < C'_{\xi, \delta} 2^{-0.005l} l^{200}$$

Since this is summable in  $l$ , the Borel Cantelli lemma completes the proof.  $\square$

### 3.3 Macroscopic time

Now we turn to macroscopic time.

**Lemma 5.** *For  $Leb_1$ -a.e.  $\lambda \in [0, 1]$ ,*

$$\lim_{\xi \rightarrow 0} \lim_{\delta' \rightarrow 0} \frac{1}{\delta'} \lim_{\varepsilon \rightarrow 0} (\mu \times \mu) \mathcal{A}_{\lambda, \frac{\delta'}{\varepsilon}, \varepsilon}^\xi = \rho(\lambda)$$

*Proof.* Let us write

$$\mathcal{A}_{\lambda, \frac{\delta'}{\varepsilon}, \varepsilon}^\xi = \cup_{k=1}^{\frac{\delta'}{\varepsilon\delta}-1} \mathcal{C}_k,$$

where  $\mathcal{C}_k = (\Phi^{-k\delta} \times \Phi^{-\lambda k\delta}) \mathcal{A}_{\lambda, \delta, \varepsilon}^\xi$ . Then the bound  $(\mu \times \mu) \mathcal{A}_{\lambda, \frac{\delta'}{\varepsilon}, \varepsilon}^\xi \leq \frac{\delta'}{\varepsilon\delta} (\mu \times \mu) \mathcal{C}_0$  and Lemma 2 give

$$\lim_{\xi \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (\mu \times \mu) \mathcal{A}_{\lambda, \frac{\delta'}{\varepsilon}, \varepsilon}^\xi \leq \delta' \rho(\lambda),$$

whence the upper bound follows.

To derive the lower bound, we write

$$(\mu \times \mu) \mathcal{A}_{\lambda, \frac{\delta'}{\varepsilon}, \varepsilon}^\xi \geq \frac{\delta'}{\varepsilon\delta} (\mu \times \mu) \mathcal{C}_0 - \sum_{0 \leq k_1 < k_2 < \frac{\delta'}{\varepsilon\delta} - 1} (\mu \times \mu) (\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2})$$

An analogous argument to the upper bound will prove the lower bound once we establish

$$\lim_{\xi \rightarrow 0} \lim_{\delta' \rightarrow 0} \frac{1}{\delta'} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \sum_{0 \leq k_1 < k_2 < \frac{\delta'}{\varepsilon\delta} - 1} (\mu \times \mu) (\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}) = 0. \quad (18)$$

First, using the invariance of  $\mu$  we have

$$\sum_{0 \leq k_1 < k_2 < \frac{\delta'}{\varepsilon\delta} - 1} (\mu \times \mu) (\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}) \leq \frac{\delta'}{\varepsilon\delta} \sum_{1 \leq k \leq \frac{\delta'}{\varepsilon\delta}} (\mu \times \mu) (\mathcal{C}_0 \cap \mathcal{C}_k). \quad (19)$$

The short returns are guaranteed to have small contribution by Lemma 4:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sum_{1 \leq k \leq (\log^{100} \varepsilon)/\delta} (\mu \times \mu)(\mathcal{C}_0 \cap \mathcal{C}_k) = 0. \quad (20)$$

To estimate the contribution of large  $k$ 's, we use Theorem 3 (actually at this point a weaker version of that theorem, namely Theorem 1.1 of [8] is enough as the initial measure is absolutely continuous.) Specifically, we fix the trajectory of the first particle (that is, we fix  $q_1, \varphi_1$ ) and for a fixed  $k$  we choose  $F = F(k)$  to be the indicator of the set  $A = A(k)$ , where

$$A = \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(\Phi^{k\delta}(q_1, \varphi_1)) = \{(q_2, \varphi_2) : (\Phi^{k\delta}(q_1, \varphi_1), q_2, \varphi_2) \in A_{\lambda, \delta, \varepsilon}^{\xi}\}$$

(recall the notation (12)). That is,  $A$  is such that there is a good collision in the time interval  $[k\delta, (k+1)\delta]$  if  $\Phi^{\lambda k\delta}(q_2, \varphi_2) \in A$ . Clearly,  $A$  is the  $\varepsilon$  neighborhood of a two dimensional surface of area  $O(\delta)$  (see the proof of Lemma 2). Thus in particular,  $F$  is generalized Lipschitz (generalized Hölder with exponent 1) with uniformly bounded norm. Now we apply Theorem 3 with  $\alpha = 1$  to conclude that

$$\begin{aligned} & (\mu \times \mu)(\mathcal{C}_0 \cap \mathcal{C}_k) \\ &= \int \mu \left( (q_2, \varphi_2) \in \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(q_1, \varphi_1), \Phi^{\lambda k\delta}(q_2, \varphi_2) \in \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(\Phi^{k\delta}(q_1, \varphi_1)) \right) d\mu(q_1, \varphi_1) \\ &\leq C \int \left[ \mu \left( \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(q_1, \varphi_1) \right) \mu \left( \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(\Phi^{k\delta}(q_1, \varphi_1)) \right) + e^{-a\sqrt{\lambda k\delta}} \right] d\mu(q_1, \varphi_1). \end{aligned}$$

Now using Lemma 3 (a) to bound  $\mu \left( \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(q_1, \varphi_1) \right)$  and  $\mu \left( \mathcal{A}_{\lambda, \delta, \varepsilon}^{\xi}(\Phi^{k\delta}(q_1, \varphi_1)) \right)$ , we conclude

$$\sum_{\frac{\log^{100} \varepsilon}{\delta} \leq k < \frac{\delta'}{\varepsilon \delta}} (\mu \times \mu)(\mathcal{C}_0 \cap \mathcal{C}_k) < C \sum_{\frac{\log^{100} \varepsilon}{\delta} \leq k < \frac{\delta'}{\varepsilon \delta}} [\varepsilon^2 \delta^2 + e^{-a\sqrt{\lambda k\delta}}].$$

This estimate, combined with (19) and (20) yields (18). We have finished the proof of Lemma 5.  $\square$

### 3.4 Proof of Theorem 1

Now we want to prove a version of Lemma 5 for a later macroscopic time interval, conditioned on the event that there has been no good collision before. As the main difficulty is the lack of independence, we apply the big block small block technique to gain approximate independence among big blocks. The big block size will be  $M = \delta'/\varepsilon$  and the small block size is  $m = \log^{100} \varepsilon$ . The  $n$ th

big block is the time interval  $[m_{n-1}, M_n]$  and the  $n$ th small block is  $[M_n, m_n]$ , where  $M_n = Mn + m(n-1)$  and  $m_n = (M+m)n$ . Let us write

$$E_n = \{ \text{there is good collision in the } n\text{th big block} \} \text{ and } D_n = \cup_{N=1}^n E_N$$

Our main proposition is

**Proposition 1.**

$$\lim_{\xi \rightarrow 0} \lim_{\delta' \rightarrow 0} \frac{1}{\delta'} \lim_{\varepsilon \rightarrow 0} (\mu \times \mu)(E_{n+1} | \overline{D_n}) = \rho(\lambda) \quad (21)$$

uniformly for  $n < T/\delta'$ .

Note that by Lemma 2,

$$(\mu \times \mu)(\text{there is good collision in some small block}) = o_\varepsilon(1).$$

and thus Theorem 1 follows from Proposition 1. It only remains to prove Proposition 1.

*Proof of Proposition 1.* We prove Proposition 1 by induction. The case  $n = 0$  is Lemma 5.

For general  $n$  we follow a similar strategy, but now the invariant measure is replaced by a Markov decomposition at time

$$\tau_n = M_n + m/2.$$

Strictly speaking, we take Markov decomposition at time  $\lfloor \tau_n/\mathfrak{s} \rfloor \mathfrak{s}$ , but for the ease of notation we simply write  $\tau_n$  (and apply similar notation for later stopping times). As before, we fix  $q_1, \varphi_1$ . For notational convenience given a set  $F$  we denote

$$F(q_1, \varphi_1) = \{(q_2, \varphi_2) : (q_1, \varphi_1, q_2, \varphi_2) \in F\}.$$

Now for the fixed  $q_1, \varphi_1$ , let  $\{\ell_{n\alpha}\}_{\alpha \in \mathfrak{A}_n}$  be the collection of the standard pairs in the image  $\Phi_*^{\tau_n} \mu(q_2, \varphi_2)$  for which there has been no good collision in the first  $n$  big blocks and  $c_{n\alpha}$  is the relative weight of the curve  $\ell_{n\alpha}$  in this family:

$$\mu(A \circ \Phi^{\tau_n} | \overline{D_n}(q_1, \varphi_1)) = \sum_{\alpha \in \mathfrak{A}_n} c_{\alpha, n} \mathbb{E}_{\ell_{n, \alpha}}(A). \quad (22)$$

Note that this Markov decomposition depends on  $q_1, \varphi_1$ .

First we claim that the contribution of such  $q_1, \varphi_1$ 's for which

$$\mu(\overline{D_n}(q_1, \varphi_1)) < \delta^2 \quad (23)$$

is negligible. To see this, first observe that by the inductive hypothesis,

$$(\mu \times \mu)(\overline{D_n}) \geq 1 - e^{-2T\rho(\lambda)}$$

holds for  $\xi$  small enough. Then writing

$$\begin{aligned} & (\mu \times \mu)(E_{n+1}|\overline{D_n}) \\ = & \frac{1}{(\mu \times \mu)\overline{D_n}} \int \mu(E_{n+1}(q_1, \varphi_1) \cap \overline{D_n}(q_1, \varphi_1)) d\mu(q_1, \varphi_1) \end{aligned} \quad (24)$$

we see that the contribution of  $(q_1, \varphi_1)$ 's satisfying (23) is bounded by  $C\delta'^2$  (from above, and by zero from below), i.e. they are indeed negligible. Let us say that  $(q_1, \varphi_1) \in G_n$  iff (23) is false. Now we want to apply the growth lemma to conclude that for  $(q_1, \varphi_1) \in G_n$ ,

$$\sum_{\alpha \in \mathfrak{A}_n \setminus \tilde{\mathfrak{A}}_n} c_{n,\alpha} < C\varepsilon^2/\delta'^2 \quad (25)$$

where

$$\tilde{\mathfrak{A}}_n = \{\alpha \in \mathfrak{A}_n : |\ell_{n,\alpha}| \geq \varepsilon^2\}.$$

Unfortunately, the growth lemma does not directly imply (25), as unstable curves may have been cut by the boundary of  $\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi$  in the past (depending on  $(q_1, \varphi_1)$ ) and such fragmentations are clearly not considered in Lemma 1. That is why we first prove

**Lemma 6.** *There is a set  $G'_n \subset G_n$  such that*

1.  $\mu(G_n \setminus G'_n) < \delta'^2$
2. *for any  $(q_1, v_1) \in G'_n$ , (25) holds.*

*Proof.* The only reason why (25) can fail to hold is that too many curves have been cut by the boundary of  $\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi$  in the first big  $n$  big blocks. Note however that by definition,  $\tau_n - M_n = m/2$  and  $|\Phi^{-m/2}W|$  is superpolynomially small in  $\varepsilon$ . Thus the curves that were cut before had to lie entirely in the  $\varepsilon^{20}$  neighborhood of the boundary of  $\mathcal{A}_{\lambda,\delta,\varepsilon}^\xi$  and the weight of such curves is small. More precisely, we define

$$\mathcal{B}_{\lambda,t,\varepsilon} = \{(q_1, \varphi_1), (q_2, \varphi_2) : \exists s \in [0, t] : ||\Pi_q \Phi^s(q_1, \varphi_1) - \Pi_q \Phi^s(q_2, \varphi_2)|| - \varepsilon| < \varepsilon^{20}\}$$

and

$$\mathcal{B}_{\lambda,t,\varepsilon}(q_1, \varphi_1) = \{(q_2, \varphi_2) : (q_1, \varphi_1, q_2, \varphi_2) \in \mathcal{B}_{\lambda,t,\varepsilon}\}.$$

First, we note that a simplified version of Lemma 2 implies

$$(\mu \times \mu)\mathcal{B}_{\lambda,T/\varepsilon,\varepsilon} < \varepsilon^{18}. \quad (26)$$

Now let us define  $G'_n \subset G_n$  by

$$(q_1, \varphi_1) \in G'_n \quad \text{iff} \quad \mu(\mathcal{B}_{\lambda, T/\varepsilon, \varepsilon}(q_1, \varphi_1)) < \varepsilon^{17}.$$

By Fubini's theorem,  $\mu(G_n \setminus G'_n) < \delta'^2$  holds. Now for a fixed  $(q_1, \varphi_1) \in G'_n$  in the Markov decomposition (22), let  $\mathfrak{A}'_n \subset \mathfrak{A}_n$  be the index set of curves  $W_{\alpha, n}$  for which there is some  $k \in [m/2\delta, \tau_n/\delta]$  such that

$$\Phi^{-k\delta} W_{\alpha, n} \quad \text{intersects} \quad \partial \mathcal{A}_{\lambda, \delta, \varepsilon}^\xi(\Phi^{\tau_n - k\delta}(q_1, \varphi_1)). \quad (27)$$

Since  $|\Phi^{-m/2} W_{\alpha, n}|$  is superpolynomially small in  $\varepsilon$ , (27) implies

$$\Phi^{-k\delta} W_{\alpha, n} \subset \mathcal{B}_{\lambda, \delta, \varepsilon}^\xi(\Phi^{\tau_n - k\delta}(q_1, \varphi_1)). \quad (28)$$

Next,  $(q_1, \varphi_1) \in G'_n$  implies

$$\sum_{\alpha \in \mathfrak{A}'_n} c_{n, \alpha} \leq \frac{1}{\mu(\overline{D_n}(q_1, \varphi_1))} \mu(\mathcal{B}_{\lambda, \tau_n - m/2, \varepsilon}(q_1, \varphi_1)) < \varepsilon^{16}.$$

Finally, if  $\alpha \in \mathfrak{A}_n \setminus \mathfrak{A}'_n$ , then  $l_{n, \alpha}$  is a full curve in the image  $\Phi^{\tau_n} \mu$  and thus by the growth lemma

$$\sum_{\alpha \in \mathfrak{A}_n \setminus \mathfrak{A}'_n, |\ell_{n, \alpha}| < \varepsilon^2} c_{n, \alpha} \leq \frac{C}{\mu(\overline{D_n}(q_1, \varphi_1))} \varepsilon^2.$$

Lemma 6 follows.  $\square$

By Lemma 6, we have

$$\left| \frac{\mu(E_{n+1}(q_1, \varphi_1) \cap \overline{D_n}(q_1, \varphi_1))}{\mu(\overline{D_n}(q_1, \varphi_1))} - \sum_{\alpha \in \mathfrak{A}_n} c_{n, \alpha} \mathbb{P}_{\ell_{n, \alpha}}(E_{n+1}) \right| < C\varepsilon^2/\delta'^2. \quad (29)$$

for any fixed  $(q_1, \varphi_1) \in G'_n$ . We conclude that

$$|(\mu \times \mu)(E_{n+1} | \overline{D_n}) - I| < C\delta'^2 + C\varepsilon^2\delta'^{-2}, \quad (30)$$

where

$$I = \frac{1}{(\mu \times \mu)\overline{D_n}} \int_{(q_1, \varphi_1) \in G'_n} \left[ \mu(\overline{D_n}(q_1, \varphi_1)) \sum_{\alpha \in \mathfrak{A}_n} c_{n, \alpha} \mathbb{P}_{\ell_{n, \alpha}}(E_{n+1}) \right] d\mu(q_1, \varphi_1). \quad (31)$$

Now we observe that Proposition 1 will be established once we prove that

$$\sum_{\alpha \in \mathfrak{A}_n} c_{n, \alpha} \mathbb{P}_{\ell_{n, \alpha}}(E_{n+1}) = \delta' \rho(\lambda)(1 + o_\xi(1)). \quad (32)$$

uniformly for  $(q_1, \varphi_1) \in G'''_n$ , where  $G'''_n \subset G'_n$  is a fixed set (to be defined later) with  $\mu(G'_n \setminus G'''_n) < \delta'^2$ . It only remains to prove (32), which is completed in the next two lemmas.

**Lemma 7. (Upper bound)**

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n,\alpha} \mathbb{P}_{\ell_{n,\alpha}}(E_{n+1}) \leq \delta' \rho(\lambda)(1 + o_\xi(1)). \quad (33)$$

for any  $(q_1, \varphi_1) \in G'_n$ .

*Proof.* First, we introduce the notation

$$\mathcal{C}'_k = \{(q_2, \varphi_2) : \Phi^{\lambda k \delta}(q_2, \varphi_2) \in \mathcal{A}_{\lambda, \delta, \varepsilon}^\xi(\Phi^{\tau_n + k \delta}(q_1, \varphi_1))\}$$

and write

$$\mathbb{P}_{\ell_{n,\alpha}}(E_{n+1}) \leq \sum_{k=\frac{m}{2\delta}}^{\frac{M+m/2}{\delta}} \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_k).$$

Now we can apply Theorem 3 (similarly to the argument in the proof of Lemma 5) to conclude

$$\mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_k) \leq \rho(\lambda)(1 + o_\xi(1))\varepsilon\delta + C\varepsilon^{-4}e^{-a\sqrt{\lambda k \delta}}, \quad (34)$$

whence (33) follows.  $\square$

**Lemma 8. (Lower bound)**

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n,\alpha} \mathbb{P}_{\ell_{n,\alpha}}(E_{n+1}) \geq \delta' \rho(\lambda)(1 + o_\xi(1)). \quad (35)$$

for all  $(q_1, \varphi_1) \in G'''_n$ .

*Proof. Step 1: Inclusion exclusion formula*

Let us introduce the notations

$$\hat{\sum}_k = \sum_{k=\frac{m}{2\delta}}^{(M+m/2)/\delta} \quad \text{and} \quad \hat{\sum}_{k_1, k_2} = \sum_{\frac{m}{2\delta} \leq k_1 < k_2 \leq (M+m/2)/\delta}.$$

Now we have the simple estimate

$$\mathbb{P}_{\ell_{n,\alpha}}(E_{n+1}) \geq \sum_k \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_k) - \sum_{k_1, k_2} \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}). \quad (36)$$

Lemma 3 (b) and Theorem 3 imply that if

$$\text{dist}(\Pi_q \Phi^{\tau_n + k \delta}(q_1, \varphi_1), \partial \mathcal{D}) > 2\xi, \quad (37)$$

then

$$\mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_k) \geq \rho(\lambda)(1 - o_\xi(1))\varepsilon\delta + C\varepsilon^{-4}e^{-a\sqrt{\lambda k \delta}}.$$



Since  $\delta \ll \xi$ , the set of  $k$ 's satisfying (37) has density  $1 - O(\sqrt{\xi})$ . (In fact, for most orbits the density is  $1 - O(\xi)$ , while  $1 - O(\sqrt{\xi})$  accommodates the orbits which are almost tangent to the boundary for many collisions). Consequently,

$$\sum_k \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_k) \geq \delta' \rho(\lambda)(1 - o_\xi(1)) + o_\varepsilon(1).$$

Now (35) would follow from the estimate

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n,\alpha} \sum_{k_1, k_2} \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) = \delta' o_\xi(1). \quad (38)$$

Unfortunately, (38) is not always true. However, we will prove that it is true for a restricted set of  $(q_1, \varphi_1)$ 's (which we denote by  $G_n'''$ ) and for most  $\alpha$ 's.

We will need the notation

$$\mathcal{K} = \left[ \frac{m}{2\delta}, (M + m/2)/\delta \right] \quad \text{and} \quad \check{\cup}_{k_2} = \cup_{k_2=k_1+1}^{k_1+m/\delta}$$

**Step 2: bound for  $k_2 - k_1 < m$**

By Lemma 4, we have for all  $k_1 \in \mathcal{K}$

$$(\mu \times \mu)(\check{\cup}_{k_2} \mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}) < \varepsilon^{1.99}.$$

and consequently

$$\sum_{k_1 \in \mathcal{K}} (\mu \times \mu)(\check{\cup}_{k_2} \mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}) < \frac{\delta'}{\varepsilon} \varepsilon^{1.99} < \varepsilon^{0.98}. \quad (39)$$

Now we say that  $\alpha \in \tilde{\tilde{\mathfrak{A}}}_n \subset \tilde{\mathfrak{A}}_n$  if

$$|\ell_{n,\alpha}| > \varepsilon^2 \text{ and } \#(\mathcal{K} \setminus \mathcal{K}'(\alpha)) < \varepsilon^{-0.5},$$

where

$$\mathcal{K}'(\alpha) = \{k_1 : \mathbb{P}_{\ell_{n,\alpha}}(\check{\cup}_{k_2} \mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) < \varepsilon^{1.1}\}. \quad (40)$$

Next, we define  $G_n''' \subset G_n'$  as the set of such  $(q_1, \varphi_1) \in G_n'$  for which

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n \setminus \tilde{\tilde{\mathfrak{A}}}_n} c_{n,\alpha} < \delta'^2. \quad (41)$$

First we claim that  $\mu(G'_n \setminus G''_n) < \delta'^2$  as needed. Assume by contradiction that  $\mu(G'_n \setminus G''_n) > \delta'^2$ . Then

$$\begin{aligned}
& \sum_{k_1 \in \mathcal{K}} (\mu \times \mu)(\check{\cup}_{k_2} \mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}) \\
& \geq \int_{(q_1, \varphi_1) \in G'_n \setminus G''_n} \sum_{k_1 \in \mathcal{K}} \mu(\check{\cup}_{k_2} \mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) d\mu(q_1, \varphi_1) \\
& \geq \int_{(q_1, \varphi_1) \in G'_n \setminus G''_n} \mu(\overline{D}_n(q_1, \varphi_1)) \sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n, \alpha} \sum_{k_1 \in \mathcal{K}} \mathbb{P}_{\ell_{n, \alpha}}(\check{\cup}_{k_2} \mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) d\mu(q_1, \varphi_1) \\
& \geq \int_{(q_1, \varphi_1) \in G'_n \setminus G''_n} \mu(\overline{D}_n(q_1, \varphi_1)) \sum_{\alpha \in \tilde{\mathfrak{A}}_n \setminus \tilde{\mathfrak{A}}'_n} c_{n, \alpha} \sum_{k_1 \in \mathcal{K} \setminus \mathcal{K}'(\alpha)} \mathbb{P}_{\ell_{n, \alpha}}(\check{\cup}_{k_2} \mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) d\mu(q_1, \varphi_1).
\end{aligned}$$

By the definition of  $G'_n$ ,  $G''_n$ ,  $\tilde{\mathfrak{A}}_n$  and  $\mathcal{K}'(\alpha)$ , we see that this last expression is bigger than  $\delta'^4 \varepsilon^{0.6}$  which is a contradiction with (39). Thus  $\mu(G'_n \setminus G''_n) < \delta'^2$  indeed holds.

For  $\alpha \in \tilde{\mathfrak{A}}_n$ , we use the estimate

$$\sum_{k_1, k_2} \mathbb{P}_{\ell_{n, \alpha}}(\mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) < \sum_{k_1} \sum_{k_2 = k_1 + m/\delta}^{(M+m)/\delta} \mathbb{P}_{\ell_{n, \alpha}}(\mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) \quad (42)$$

$$+ \log^{100} \varepsilon \sum_{k_1 \in \mathcal{K}'(\alpha)} \mathbb{P}_{\ell_{n, \alpha}}(\check{\cup}_{k_2} \mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) \quad (43)$$

$$+ \log^{100} \varepsilon \sum_{k_1 \in \mathcal{K} \setminus \mathcal{K}'(\alpha)} \mathbb{P}_{\ell_{n, \alpha}}(\mathcal{C}'_{k_1}). \quad (44)$$

By the definition of  $\mathcal{K}'(\alpha)$ , (43) is bounded by  $\varepsilon^{0.05}$ . Since  $\alpha \in \tilde{\mathfrak{A}}_n$  and by Theorem 3, (44) is bounded by  $\varepsilon^{0.4}$ .

### Step 3: bound for $k_2 - k_1 \geq m$

In order to estimate (42), we use Markov decomposition at time  $\tau_{n, k_1} := \tau_n + k_1 \delta + \frac{m}{2}$  conditioned on  $\mathcal{C}'_{k_1}$

$$\mathbb{P}_{\ell_{n, \alpha}}(A \circ \Phi^{k_1 + m/2} | \mathcal{C}'_{k_1}) = \sum_{\beta \in \mathfrak{B}_{n, \alpha, k_1}} c_{n, \alpha, k_1, \beta} \mathbb{E}_{\ell_{n, \alpha, k_1, \beta}}(A). \quad (45)$$

By (34), we have  $\mathbb{P}_{\ell_{n, \alpha}}(\mathcal{C}'_{k_1}) < C \delta \varepsilon$ . Now we want to guarantee that the short curves in  $\mathfrak{B}_{n, \alpha, k_1}$  have small weight, at least for most  $\alpha$ 's.

Using (26), we see that

$$\int_{(q_1, \varphi_1) \in G''_n} \sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n, \alpha} \mathbb{P}_{\ell_{n, \alpha}}(\mathcal{D}'_{k_1}) d\mu(q_1, v_1) < \varepsilon^{17},$$

where  $\mathcal{D}'$  is defined as  $\mathcal{C}'$  with  $\mathcal{A}$  replaced by  $\mathcal{B}$ . Now we define  $G_{n,k_1}''' \subset G_n''$  as the set of  $(q_1, \varphi_1)$ 's for which

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n,\alpha} \mathbb{P}_{\ell_{n\alpha}}(\mathcal{D}'_{k_1}) < \varepsilon^{14}.$$

and  $G_n''' = \cap_{k_1} G_{n,k_1}'''$ . By Fubini's theorem,  $\mu(G_n'' \setminus G_n''') < \varepsilon$ . From now on, we assume  $(q_1, \varphi_1) \in G_n'''$ .

Next, we define  $\tilde{\mathfrak{A}}_n \subset \tilde{\mathfrak{A}}_n$  as the set of such  $\alpha$ 's for which

$$\sum_{k_1} \mathbb{P}_{\ell_{n\alpha}}(\mathcal{D}'_{k_1}) < \varepsilon^{11}.$$

Again by Fubini's theorem,

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n \setminus \tilde{\mathfrak{A}}_n} c_{n,\alpha} < \varepsilon. \quad (46)$$

Now we can repeat the second half of the proof of Lemma 6 to conclude that for  $(q_1, \varphi_1) \in G_n'''$  and for  $\alpha \in \tilde{\mathfrak{A}}_n$ ,

$$\sum_{\beta \in \mathfrak{B}_{n,\alpha,k_1}, |\ell_{n,\alpha,k_1,\beta}| < \varepsilon^{-6}} c_{n,\alpha,k_1,\beta} < \varepsilon^{-4}.$$

Now if  $|\ell_{n,\alpha,k_1,\beta}| > \varepsilon^{-6}$ , we use the same argument as in (34) to conclude

$$\mathbb{P}_{\ell_{n,\alpha,k_1,\beta}}(\mathcal{C}_{k_2-k_1-m/(2\delta)}'') < C\delta\varepsilon,$$

where

$$\mathcal{C}_k'' = \{(q_2, \varphi_2) : \Phi^{\lambda k\delta}(q_2, \varphi_2) \in \mathcal{A}_{\lambda,\delta,\varepsilon}^\xi(\Phi^{\tau_{n,k_1}+k\delta}(q_1, \varphi_1))\}.$$

Hence for  $\alpha \in \tilde{\mathfrak{A}}_n$ , (42) is bounded by

$$\sum_{k_1} \mathbb{P}_{\ell_{n\alpha}}(\mathcal{C}'_{k_1}) \sum_{k_2=k_1+m/\delta}^{(M+m)/\delta} \sum_{\beta \in \mathfrak{B}_{n,\alpha,k_1}} c_{n,\alpha,k_1,\beta} \mathbb{P}_{\ell_{n,\alpha,k_1,\beta}}(\mathcal{C}_{k_2-k_1-m/(2\delta)}'') < C\delta'^2.$$

We conclude

$$\sum_{\alpha \in \tilde{\mathfrak{A}}_n} c_{n,\alpha} \sum_{k_1, k_2} \mathbb{P}_{\ell_{n,\alpha}}(\mathcal{C}'_{k_1} \cap \mathcal{C}'_{k_2}) = \delta' o_\xi(1). \quad (47)$$

#### Step 4: Finishing the proof

By (41) and (46), we can replace (38) in Step 1 by (47). Lemma 8 follows. We have finished the proof of Proposition 1 and Theorem 1.  $\square$

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